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Summing tree graphs at threshold

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The solution of the classical field equation generates the sum of all tree graphs. We show that the classical equation reduces to an easily solved ordinary differential equation for certain multiparticle threshold amplitudes and compute these amplitudes.

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The important and outstanding problem of high-energy baryon-number violation [1] motivates the investigation of simpler processes which share some of its features. One feature of the high-energy baryon violation is the production of a very large number of particles. Even in a weakly coupled theory, many-particle amplitudes may become large because they involve a large number of graphs. Cornwall [2] and Goldberg [3] have examined the lowest-order, tree-graph amplitudes for many-particle production. Recently, Voloshin [4] has considered the tree graphs in the simple, unbroken $\lambda\,\phi^4$ theory for the amplitude where a highly off-mass-shell ϕ field produces a large number n of on-mass-shell ϕ particles. In particular, he considered the threshold limit in which all the produced particles are at rest and obtained an exact result for this amplitude by deriving and solving a recursion relation for the many-particle amplitudes. Using this technique, Argyres, Kleiss, and Papadopoulos [5] extended Voloshin's result to include the case of the spontaneously broken $\lambda\,\phi^4$ theory which contains cubic as well as quartic interactions. The purpose of this Rapid Communication is to point out that these previous results are obtained very simply if one recalls that the generating function for the tree graphs is the solution to the classical field equation [6]. We shall also show how to generalize the results to the case of the unbroken, multicomponent O(N) scalar theory.

Our object is to compute the amplitudes for the field ϕ to create n particles out of the vacuum, $\langle n|\phi|0\rangle$, for the simple scalar theory described by the Lagrange function

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \rho\phi\,. \tag{1}$$

The source ρ is introduced so as to generate these amplitudes according to the usual reduction formula method [7],

$$\langle n|\phi(x)|0\rangle = \prod_{a=1}^{n} \int (d^4x_a)\, e^{-ip_a x_a}\,(p_a^2+m^2)\;\frac{\delta}{\delta\rho(x_a)}\langle 0+|\phi(x)|0-\rangle^{\rho}\Big|_{\rho=0}\,. \tag{2}$$

The tree-graph approximation is obtained by the replacement

$$\langle 0+|\phi(x)|0-\rangle^{\rho} \to \phi_{\rm cl}(x)\,, \tag{3}$$

where $\phi_{\rm cl}$ is the solution to the classical field equation driven by the source ρ,

$$(-\partial^2+m^2)\phi_{\rm cl} + \frac{1}{3!}\lambda\phi_{\rm cl}^3 = \rho\,, \tag{4}$$

subject to the quantum time-ordered boundary conditions which give the prescription $m^2 \to m^2 - i\epsilon$ in the propagator. This defines the classical field $\phi_{\rm cl}$ as a functional of the source, $\phi_{\rm cl} = \phi_{\rm cl}[\rho]$. We shall consider only the threshold limit $\mathbf{p}_a = 0$. In this limit, the space-time-dependent source $\rho(x)$ and the resulting field $\phi_{\rm cl}(x)$ may be replaced by spatially uniform but time-dependent functions $\rho(t)$ and $\phi_{\rm cl}(t)$. Thus the field equation (4) reduces to an ordinary differential equation. The mass-shell amplitude is obtained by setting

$$\rho(t) = \rho_0\, e^{i\omega t} \tag{5}$$

and then taking the limit $\omega \to m$. To see how this goes, we first examine the solution to the classical field equation (4) when there is no interaction ($\lambda \to 0$),

$$\phi_{\rm cl} \to z(t) \equiv z_0\, e^{i\omega t}\,, \tag{6}$$

where

$$z_0 = \frac{\rho_0}{m^2 - i\epsilon - \omega^2}\,. \tag{7}$$

We now insert the expansion of $\phi_{\rm cl}$ in powers of the coupling λ

$$\phi_{\rm cl} = z + \lambda\,\phi_{\rm cl}^{(1)} + \lambda^2\,\phi_{\rm cl}^{(2)} + \cdots \tag{8}$$

into the field equation (4) and identify the coefficients of the various powers of the coupling λ. This shows that $\phi_{\rm cl}^{(1)}$ is proportional to $\lambda\, z^3/3!$. Similarly, proceeding to higher orders in the expansion shows that $\phi_{\rm cl}$ is an ordinary function of $z(t)$, $\phi_{\rm cl}(t) = \phi_{\rm cl}(z(t))$. Hence, in view of Eqs. (6) and (7), the functional derivatives which occur in the reduction formula (2) become ordinary derivatives:

$$\int (d^4x_a)\, e^{i\omega t_a}\, (m^2-\omega^2)\, \frac{\delta}{\delta\rho(x_a)} \phi_{\rm cl}(t;\,[\rho]) = \frac{\partial}{\partial z_0} \phi_{\rm cl}(z(t))\,. \quad (9)$$

We see that the threshold n-particle amplitude in the tree approximation may be expressed as

$$\langle n|\phi(0)|0\rangle^{\rm tree}_{\rm threshold} = \left(\frac{\partial}{\partial z}\right)^n \phi_{\rm cl}\bigg|_{z=0}\,. \quad (10)$$

To go on mass shell, $\omega \to m$, we take $\rho_0 \to 0$ in such a way as to keep $z(t)$ finite. In this limit, the classical field obeys the homogeneous, ordinary differential equation

$$\left[\frac{d^2}{dt^2} + m^2\right] \phi_{\rm cl}(t) + \frac{1}{3!}\lambda\phi^3_{\rm cl}(t) = 0\,, \quad (11)$$

with the condition that $\phi_{\rm cl}(t)$ approaches $z(t)$ as λ vanishes.

A. Unbroken symmetry

The ordinary differential equation (11) has a constant of the motion, the energy integral

$$\frac{1}{2}\left(\frac{d\phi_{\rm cl}}{dt}\right)^2 + \frac{1}{2}m^2\,\phi^2_{\rm cl} + \frac{1}{4!}\lambda\,\phi^4_{\rm cl} = E^2\,. \quad (12)$$

Since $\phi_{\rm cl}$ contains only an ascending power series in the oscillating function $z(t)$, the left-hand side of this equation contains only oscillating terms. Hence the constant E must vanish, and the final integration to obtain $\phi_{\rm cl}$ gives a simple elementary function. Rather than indicating the intermediate steps, we shall just write down the result since it is easier to verify directly that it is the proper solution of the original differential equation (11):

$$\phi_{\rm cl}(t) = \frac{z(t)}{1-(\lambda/48\,m^2)z(t)^2}\,. \quad (13)$$

This is the function for the case of unbroken symmetry with $m^2 > 0$. Placing the solution (13) in Eq. (10) shows that the amplitude vanishes unless $n = 2k+1$ is odd, where

$$\langle 2k+1|\phi(0)|0\rangle^{\rm tree}_{\rm threshold} = (2k+1)!\left(\frac{\lambda}{48\,m^2}\right)^k\,. \quad (14)$$

This is the result of Voloshin [4].

B. Broken symmetry

The reflection symmetry $\phi \to -\phi$ is broken when $m^2 \to -m^2 < 0$. In this case the field equation (11) has the constant solution

$$\phi_{\rm cl} \to \phi_0 = \sqrt{\frac{3!\,m^2}{\lambda}}\,. \quad (15)$$

Expanding the field about this constant solution gives rise to three-field as well as four-field couplings, and the graphical structure becomes more complex. Moreover, the mass parameter in the Lagrange function for the shifted field is altered to $m_1 = \sqrt{2}\,m$ and so we must now expand about

$$z(t) = z_0\, e^{im_1 t}\,. \quad (16)$$

Again since an energy integral to the equation of motion exists, the classical field equation (11) may be solved. Rather than writing down the intermediate steps, it again suffices to display the solution since its verification is simple:

$$\phi_{\rm cl}(t) = \phi_0 \frac{1+z(t)/2\phi_0}{1-z(t)/2\phi_0}\,. \quad (17)$$

Inserting this solution into Eq. (10) gives

$$\langle n|\phi(0)|0\rangle^{\rm tree}_{\rm threshold} = n!\left(\frac{1}{2\phi_0}\right)^{n-1}\,. \quad (18)$$

This is the result of Argyres *et al.* [5].

C. O(N) Model

The results of the unbroken theory are easily extended to the O(N) theory in which the single ϕ field is replaced by a vector field ϕ^a with N components, and the interaction now involves the O(N) invariant $(\phi^a\phi^a)^2$. It is straightforward to check that the previous solution (13) generalizes to

$$\phi^a_{\rm cl}(t) = \frac{z^a(t)}{1-(\lambda/48\,m^2)z(t)\cdot z(t)}\,. \quad (19)$$

In the previous simple one-component theory, the squared matrix element, divided by the Bose symmetrization factor $n!$ and multiplied by the appropriate phase-space factor, gives the threshold limit of the absorptive part of the ϕ-field propagator. The analogous construction for the O(N) theory involves some combinatorial analysis which, as we now show, is simplified by using functional methods. For the O(N) theory, the absorptive part of the propagator in the tree approximation entails the schematic structure

$$M^{ab}(2k+1) = \sum \langle 0|\phi^a|2k+1\rangle\langle 2k+1|\phi^b|0\rangle\bigg|^{\rm tree}_{\rm threshold}\,. \quad (20)$$

Here the sum is over all states with the total particle number $n = 2k+1$ including implicit Bose symmetrization factors. To perform this sum with the correct factors, we note that the exponential operation applied to a squared matrix element in our z representation,

$$\exp\left\{\frac{\partial}{\partial z^{*a}}\frac{\partial}{\partial z^a}\right\} = \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{\partial}{\partial z^{*a}}\right)^l\left(\frac{\partial}{\partial z^a}\right)^l \quad (21)$$

(with no sum over a), produces a sum over all particle numbers l of particle type a in the resulting intermediate state with the correct Bose symmetrization factor of $l!$ in the denominator. Hence the operation

$$\exp\left\{\frac{\partial}{\partial z^{*1}}\frac{\partial}{\partial z^{1}}\right\}\cdots\exp\left\{\frac{\partial}{\partial z^{*N}}\frac{\partial}{\partial z^{N}}\right\} = \exp\left\{\frac{\partial}{\partial z^{*}}\cdot\frac{\partial}{\partial z}\right\} \tag{22}$$

applied to the monomial

$$\left(\frac{\lambda}{48\,m^2}\right)^{2k} z^{*a}(z^*\cdot z^*)^k z^b (z\cdot z)^k \,,$$

which is the order $2k+1$ term in the expansion of $\phi^{*a}_{\rm cl}\,\phi^{b}_{\rm cl}$, produces the sum (20) in the limit $z^{*a} = z^b = 0$. Since

$$\exp\left\{y\frac{\partial}{\partial x}\right\} f(x) = f(x+y)\,, \tag{23}$$

we have

$$M^{ab}(2k+1) = \left(\frac{\lambda}{48\,m^2}\right)^{2k} \frac{\partial}{\partial z^a}\left(\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial z}\right)^k z^b(z\cdot z)^k \,. \tag{24}$$

The derivatives that appear here may be evaluated recursively by using

$$\left(\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial z}\right) z^a(z\cdot z)^l = 4l\,(l+N/2)\,z^a(z\cdot z)^{l-1} \,. \tag{25}$$

Thus

$$M^{ab}(2k+1) = \delta^{ab}\left(\frac{\lambda}{48\,m^2}\right)^{2k} 4^k k!\,\frac{\Gamma(k+1+N/2)}{\Gamma(1+N/2)} \,. \tag{26}$$

D. Quantum mechanics

The validity of the tree approximation is illustrated by considering the quantum mechanical analogue of the field theory, the field theory in zero spatial dimensions, the anharmonic oscillator. In this case, the "threshold limit" gives the full amplitude. Dividing the previous result for the unbroken theory (14) by $\sqrt{(2k+1)!(2m)^{2k+1}}$ to produce the normalized quantum mechanical amplitude and then squaring to obtain the spectral weight gives

$$r^{\rm tree}_{2k+1} = \left|\langle 2k+1|q|0\rangle^{\rm tree}\right|^2 = \frac{1}{2m}\left(\frac{\lambda}{96\,m^3}\right)^{2k}(2k+1)!\,. \tag{27}$$

Although the coupling λ may be small, this amplitude becomes large for a sufficiently highly excited state where k is very large, and the tree approximation must break down. Indeed, inserting the complete set of intermediate energy eigenstates into the ground-state commutator matrix element

$$\langle 0|[q,\,dq/dt]|0\rangle = i \tag{28}$$

yields the sum rule

$$\sum_n 2E_n\, r_n = 1\,, \tag{29}$$

where we have chosen the energy scale to make the ground-state energy vanish. Since a sum of positive terms appears here, and since the anharmonic coupling increases the energy of an intermediate state, we have the bound[1]

$$r_n < \frac{1}{2E_n} < \frac{1}{2m}\frac{1}{n}\,. \tag{30}$$

Hence, even for weak coupling, the tree approximation must break down for highly excited states where

$$(2k+1)(2k+1)! \sim \left(\frac{96\,m^3}{\lambda}\right)^{2k}\,, \tag{31}$$

or, on using Stirling's approximation for the factorial, when

$$k \sim (48\,m^3/\lambda)\,e\,, \tag{32}$$

where e is the base of the natural logarithm. This restriction just states that the tree approximation must break down for states which are so highly excited that the anharmonic interaction becomes comparable to the harmonic term, $m^2\langle q^2\rangle \sim \lambda\,\langle q^4\rangle$ since, in the harmonic approximation, $\langle q^2\rangle \sim k/m$ and $\langle q^4\rangle \sim k^2/m^2$.

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[1]The inequality (30) can obviously be sharpened by placing a factor of $(1-2E_1\,r_1)$ on the right-hand side. In the weak coupling limit, this factor is of order $(\lambda/m^3)^2$. This, however, does not alter the leading behavior of the large k restriction shown in Eq. (32).

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